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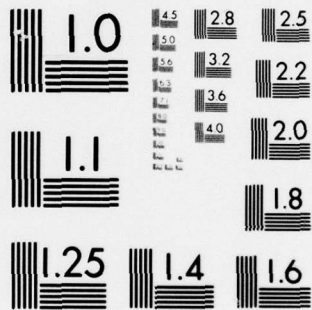
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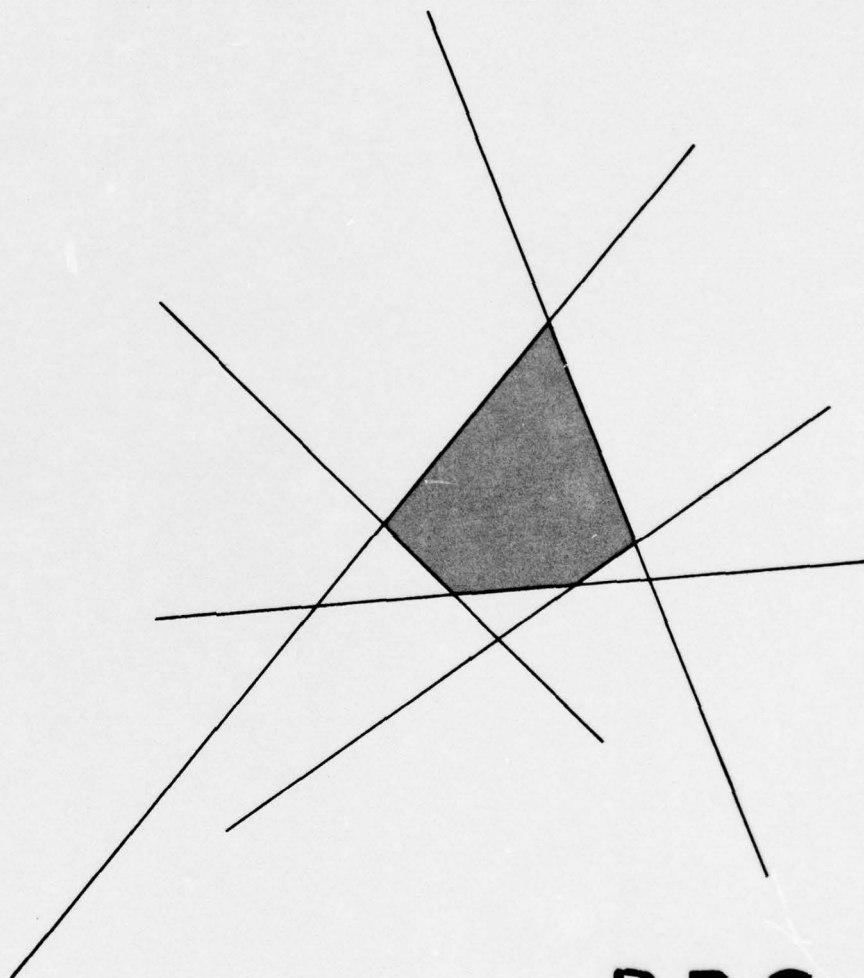
# COHERENT SYSTEMS WITH MULTI-STATE COMPONENTS

by

RICHARD E. BARLOW

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COHERENT SYSTEMS WITH MULTI-STATE COMPONENTS

by

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# ABSTRACT

The theory of binary coherent systems is generalized for multi-state components. The system state is defined to be the state of the "worst" component in the "best" min path, or equivalently, the state of the "best" component in the "worst" min cut. All of the results for the binary case can be computed for multi-state systems using the binary structure and reliability function concepts. Monotonicity results are now valid with respect to stochastic ordering of component probability vectors.

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# COHERENT SYSTEMS WITH MULTI-STATE COMPONENTS

by

R. E. Barlow

## 1. INTRODUCTION

The theory of binary coherent structures has served as a unifying foundation for a mathematical theory of reliability [1]. Various generalizations to multi-state coherent structures have been suggested ([2],[3],[4]). However, these generalizations have not been fruitful except for very special applications. We define a system state function for coherent systems with multi-state components and investigate its properties. Many results for the binary case have natural extensions in terms of this system state function. These results also have applications in fault tree analysis.

Suppose that we have a system with components  $C = \{1, 2, \dots, n\}$ . Furthermore, suppose that each component can be in one of  $m + 1$  states,  $\{0, 1, 2, \dots, m\}$ , where 0 is the failed state and  $m$  is the maximal or "perfect" state. In addition, we have given sets of components called min path sets  $\{P_1, P_2, \dots, P_p\}$  where  $C = \bigcup_{r=1}^p P_r$ . No min path is properly contained in any other min path set. The components,  $C$ , together with the min path sets define a coherent system. The system structure determines the min path sets. Intuitively, if all components in at least one min path set are "functioning", then the system is "functioning". This is the set theoretic definition of a coherent system [cf. [5]]. The min path sets determine a blocking collection of sets called the min cut sets  $\{K_1, K_2, \dots, K_k\}$ . Each min cut set meets each min path set and  $C = \bigcup_{s=1}^k K_s$ . Also, no min cut set is properly contained in any other min cut set. Intuitively, if all components in at least one min cut set are "not functioning", then the system is "not functioning". Either the min path sets or the min cut sets uniquely

determine our coherent system.

Let  $z_i = j$  if component  $i$  is in state  $j$  ( $0 \leq j \leq m$ ) so that  $\underline{z} = (z_1, z_2, \dots, z_n)$  is the component state vector. The specification and determination of component states will in general depend on engineering and system considerations which we will not discuss here. Let capital  $Z_i$  be a random variable and let  $P(Z_i = j) = p_{ij} \geq 0$  where  $\sum_{j=0}^m p_{ij} = 1$ . Since  $p_{ij}$  could be 0 for some states, it is not necessary that every component be capable of assuming every state. In general, component states will be qualitative measures as are the concepts "failed" and "functioning". In much of what follows, it is not necessary to confine component states to integer values  $\{0, 1, 2, \dots, m\}$ .  $Z_i$  could, for example, take values in  $[0, 1]$  or even negative values. However, we do not pursue this generalization here.

The performance level of a system, given the component state vector  $\underline{z} = (z_1, z_2, \dots, z_n)$ , will be system dependent and it is unlikely that any one mathematical definition of system performance will be preferred above all others. Hence, we concentrate on a fundamental, but necessarily limited measure of system performance. If the coherent system is a series system, then we assign to the system the state of its "worst" component; i.e. if  $\zeta$  is the system state function, then  $\zeta(\underline{z}) = \min_{1 \leq i \leq n} z_i$ . Intuitively, a series system is no better than its worst component. If the coherent system is a parallel system, then we assign to the system the state of its "best" components; i.e.  $\zeta(\underline{z}) = \max_{1 \leq i \leq n} z_i$ . We will make use of the following well known

### 1.1 Proposition:

For a coherent system with min path sets  $\{P_1, P_2, \dots, P_p\}$  and min cut sets  $\{K_1, K_2, \dots, K_k\}$  and any real valued function  $f_i$ ,



$$\max_{1 \leq r \leq p} \min_{i \in P_r} f_i = \min_{1 \leq s \leq k} \max_{i \in K_s} f_i.$$

[See [1], p. 12 and 15.]

### 1.1 Definition:

For a coherent system with min path sets  $\{P_1, P_2, \dots, P_p\}$  and min cut sets  $\{K_1, K_2, \dots, K_k\}$  the *system state function* is

$$(1.1) \quad \zeta(\underline{z}) = \max_{1 \leq r \leq p} \min_{i \in P_r} z_i = \min_{1 \leq s \leq k} \max_{i \in K_s} z_i.$$

Note that  $\zeta(\underline{z})$  is coordinate-wise nondecreasing. Intuitively,  $\zeta(\underline{z})$  is the state of the "worst" component in the "best" min path set or, equivalently, the state of the "best" component in the "worst" min cut set.

With this definition of system state, most of the results for binary coherent systems have a natural generalization. For example, suppose that  $t_{ij}$  is the first time that component  $i$  reaches state  $j$  starting in state  $m$ , then the time until the system first reaches state  $j$  starting in state  $m$ ,  $\tau_j$ , is easily seen to be

$$(1.2) \quad \tau_j = \max_{1 \leq r \leq p} \min_{i \in P_r} t_{ij} = \min_{1 \leq s \leq k} \max_{i \in K_s} t_{ij}.$$

The result that redundancy at the component level is better than redundancy at the system level, [Theorem 2.4, p. 8 [1]], also has a natural generalization. Let  $\underline{z} = (z_1, z_2, \dots, z_n)$  and  $\underline{w} = (w_1, w_2, \dots, w_n)$  be component state vectors for components  $\{1, 2, \dots, n\}$  and  $\{1^1, 2^1, \dots, n^1\}$  respectively. Components  $i$  and  $i^1$  may be identical but operate independently of each other.

Define  $\underline{z} \vee \underline{w} = (z_1 \vee w_1, z_2 \vee w_2, \dots, z_n \vee w_n)$

where  $z_i \vee w_i = \max(z_i, w_i)$  . Then

$$(1.3) \quad \zeta(\underline{z} \vee \underline{w}) \geq \zeta(\underline{z}) \vee \zeta(\underline{w}) .$$

This is an immediate consequence of the coordinatwise nondecreasing property of  $\zeta$  . [Definition 1.1.]

## 2. STOCHASTIC PROPERTIES OF THE SYSTEM STATE FUNCTION

Let

$$x_{ij} = \begin{cases} 1 & \text{if component } i \text{ is in state } j \\ 0 & \text{otherwise.} \end{cases}$$

Clearly  $\sum_{j=0}^m x_{ij} = 1$ . Let  $\phi$  be the usual coherent structure function associated with min path sets  $\{P_1, P_2, \dots, P_p\}$ . Then  $\phi(x_{1m}, x_{2m}, \dots, x_{mm}) = 1$  if and only if at least one min path set has all of its components in state  $m$ . Also  $\phi(x_{1m}, x_{2m}, \dots, x_{nm}) = 0$  if and only if at least one min cut set has no components in state  $m$ . Thus  $\phi$  is the coherent structure function which recognizes only two states: the state  $m$  (or perfect state) and the nonperfect set of states less than  $m$ .

Let  $y_{ij} = \sum_{r=j}^m x_{ir}$  and  $\underline{y}_j = (y_{1j}, y_{2j}, \dots, y_{nj})$ . Clearly  $\underline{y}_j \geq \underline{y}_{j+1}$  coordinatewise, so that

$$\phi(\underline{y}_j) - \phi(\underline{y}_{j+1}) \geq 0.$$

Also  $\phi(\underline{y}_j) = 1$  if and only if there is at least one min path all of whose components are in state  $j$  or greater. Hence  $\zeta(\underline{z}) \geq j$  if and only if  $\phi(\underline{y}_j) = 1$ . It follows that

$$(2.1) \quad \phi(\underline{y}_j) - \phi(\underline{y}_{j+1}) = \begin{cases} 1 & \text{if } \zeta(\underline{z}) = j \\ 0 & \text{otherwise.} \end{cases}$$

This observation will be helpful for computing  $P(\zeta(\underline{Z}) = j)$  since

$$(2.2) \quad \begin{aligned} P(\zeta(\underline{Z}) = j) &= P[\phi(\underline{Y}_j) - \phi(\underline{Y}_{j+1}) = 1] \\ &= E\phi(\underline{Y}_j) - E\phi(\underline{Y}_{j+1}). \end{aligned}$$

In the following we assume that all components are statistically independent.

We have proved the following

2.1 Theorem:

Let  $P(X_{ij} = 1) = p_{ij}$  and  $q_{ij} = \sum_{r=j}^m p_{ir}$ . For a coherent system with structure function  $\phi$  and reliability function

$$h(\underline{p}_m) = E\phi(X_{1m}, X_{2m}, \dots, X_{nm})$$

we have

$$(2.3) \quad P(\zeta(\underline{Z}) = j) = h(\underline{q}_j) - h(\underline{q}_{j+1}) \quad 0 \leq j \leq m-1$$

$$P(\zeta(\underline{Z}) = m) = h(\underline{q}_m).$$

$$(2.4) \quad P(\zeta(\underline{Z}) \geq j) = h(\underline{q}_j) \quad 0 \leq j \leq m,$$

where  $\underline{q}_j = (q_{1j}, q_{2j}, \dots, q_{nj})$ .

Example:

Consider the following two terminal network representation for a three component coherent system. Assume components can be in any one of three states: 0 for failed, 1 for marginal and 2 for perfect. The usual

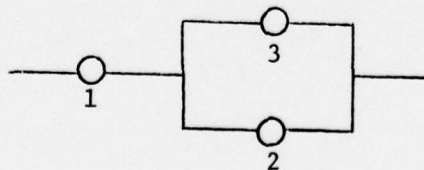


FIGURE 2.1. Two Terminal Network.

reliability function is

$$h(\underline{p}_2) = p_{12}(p_{22} \parallel p_{32})$$



where  $p_{22} \parallel p_{32} = p_{22} + p_{32} - p_{22}p_{32}$ . Let  $q_{i1} = p_{i1} + p_{i2}$  and  $q_{i2} = p_{i2}$  for  $i \leq i \leq 3$ . Then

$$P(\text{system is marginal}) = P(\zeta = 1)$$

$$= h(q_1) - h(q_2)$$

$$= q_{11}(q_{21} \parallel q_{31}) - q_{12}(q_{22} \parallel q_{32}) .$$

#### Notation:

It will be convenient to let  $\underline{p} = (p_{ij})$  be the probability matrix corresponding to component state matrix  $\underline{x} = (x_{ij})$ . Let

$$(2.5) \quad h_j(\underline{p}) = h(q_j) - h(q_{j+1}) \quad 0 \leq j \leq m-1$$

and

$$h_m(\underline{p}) = h(q_m)$$

be the probability that the coherent system is in state  $j$  and let  $\underline{h}(\underline{p}) = (h_0(\underline{p}), h_1(\underline{p}), \dots, h_m(\underline{p}))$ . The usual monotonicity properties of binary coherent systems have analogous monotonicity properties with respect to *stochastic ordering*. Let  $\underline{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_m)$  be a probability vector; i.e.  $0 \leq \alpha_j \leq 1$  and  $\sum_{j=0}^m \alpha_j = 1$ .

#### Definition:

$$\underline{\alpha} \underset{\text{st}}{\leq} \underline{\alpha}^* \quad \text{iff}$$

$$\sum_{r=j}^m \alpha_r \leq \sum_{r=j}^m \alpha_r^* \quad \text{for} \quad 0 \leq j \leq m ;$$

i.e.  $\underline{\alpha}$  is stochastically less than  $\underline{\alpha}^*$ .



Definition:

$$\underline{p} \underset{\text{st}}{\leq} \underline{p}^* \quad \text{iff}$$

$$(p_{i0}, p_{i1}, \dots, p_{im}) \underset{\text{st}}{\leq} (p_{i0}^*, p_{i1}^*, \dots, p_{im}^*)$$

for  $i \leq i \leq n$ . Intuitively, the better the components, the better the system state. More precisely, we have

2.2 Proposition:

If  $\underline{p} \underset{\text{st}}{\leq} \underline{p}^*$ , then

$$(2.6) \quad h(\underline{p}) \underset{\text{st}}{\leq} h(\underline{p}^*) .$$

Proof:

To show  $\sum_{r=j}^m h_r(\underline{p}) \leq \sum_{r=j}^m h_r(\underline{p}^*)$  we need only verify that  $h(q_j) \leq h(q_j^*)$  by Theorem 2.1 (2.4). Since  $\underline{p} \underset{\text{st}}{\leq} \underline{p}^*$  implies  $q_{ij} = \sum_{r=j}^m p_{ir} \leq \sum_{r=j}^m p_{ir}^* = q_{ij}^*$  and  $h$  is coordinatewise nondecreasing, the result is obvious. ||

Generalization of the Moore-Shannon Theorem

Moore and Shannon showed that binary coherent reliability functions are S-shaped in the sense that if all components work with probability  $p$ , either  $h(p) \geq p$  or  $h(p) \leq p$  for all  $p$ , or there exists  $p_0$  such that  $h(p) \leq p$  for  $p \leq p_0$  while  $h(p) \geq p$  for  $p \geq p_0$ . [Cf. Theorem 5.4 [1].] This result, comparing an arbitrary binary coherent system reliability with a single component reliability, has a natural generalization with respect to stochastic ordering.

### 2.3 Proposition:

Let  $(p_{i0}, p_{i1}, \dots, p_{im}) = (\alpha_0, \alpha_1, \dots, \alpha_m) = \underline{\alpha}$ . Assume  $h(p_0) = p_0$  ( $0 < p_0 < 1$ ) is the fixed point for the corresponding binary coherent reliability function. Let  $\underline{\alpha}^* = (1 - p_0, 0, \dots, 0, p_0)$ . Then

$$(a) \quad \underline{\alpha} \underset{st}{\leq} \underline{\alpha}^* \text{ implies } \underline{h(\underline{\alpha})} \underset{st}{\leq} \underline{\alpha}, \text{ while}$$

$$(b) \quad \underline{\alpha} \underset{st}{\geq} \underline{\alpha}^* \text{ implies } \underline{h(\underline{\alpha})} \underset{st}{\geq} \underline{\alpha}$$

where  $\underline{\alpha}$  is the probability matrix with identical rows,  $\underline{\alpha}$ .

#### Proof:

To show (a) we need only verify

$$\sum_{r=j}^m h_r(\underline{\alpha}) \leq \sum_{r=j}^m h_r(\underline{\alpha}^*) .$$

But  $\sum_{r=j}^m h_r(\underline{\alpha}) = h(\underline{q}_j) = h\left(\sum_{r=j}^m \alpha_r\right)$  and  $\sum_{r=j}^m \alpha_r \leq \sum_{r=j}^m \alpha_r^* = p_0$  for  $1 \leq j \leq m$

by assumption. By the Moore-Shannon Theorem,  $h\left(\sum_{r=j}^m \alpha_r\right) \leq \sum_{r=j}^m \alpha_r$  for

$0 \leq j \leq m$  which in turn implies  $\underline{h(\underline{\alpha})} \underset{st}{\leq} \underline{\alpha}$ .

(b) is proved similarly. ||

Proposition 2.3 allows us to compare arbitrary coherent systems (with identical components) to a one component system with the *same* probability vector. Of course, if no  $p_0$  ( $0 < p_0 < 1$ ) such that  $h(p_0) = p_0$  exists, then either  $\underline{h(\underline{\alpha})} \underset{st}{\leq} \underline{\alpha}$  or  $\underline{h(\underline{\alpha})} \underset{st}{\geq} \underline{\alpha}$  for all  $\underline{\alpha}$ .

### A Geometric Property of the System State Distribution

A fundamental result in coherent structure theory is that the distribution of time to first system failure is IFRA (for increasing failure rate average) if component life distributions have the same property. The same kind of result carries over to the system state distribution.

Suppose states  $\{j, j+1, \dots, m\}$  are the "good" states and the distribution of time for a component to leave the good states starting in state  $m$  is IFRA; i.e.  $\{P[Z_1(t) \geq j]\}^{1/t}$  is nonincreasing in  $t \geq 0$  for fixed  $j$ . Then

$$\{P[\zeta(t) \geq j]\}^{1/t}$$

is also nonincreasing in  $t \geq 0$  for fixed  $j$ . This is the so-called IFRA closure theorem [Cf. Theorem 2.6, [1]]. The following is a corollary to the IFRA closure theorem.

#### 2.4 Proposition:

If  $\{P[Z_1(t) \geq j]\}^{1/j}$  is nonincreasing in  $j \geq 0$  for fixed  $t$ , then

$$\{P[\zeta(t) \geq j]\}^{1/j}$$

is also nonincreasing in  $j \geq 0$  for fixed  $t$ .

#### Proof:

$$P[\zeta(t) \geq j] = h[q_j(t)] \text{ where}$$

$$q_{ij}(t) = P[Z_i(t) \geq j].$$

Define  $\bar{F}_i(j) = \sum_{r=j}^m p_{ir}(t)$  and elsewhere by constructing the linear interpolate to the points  $(j, -\log \bar{F}_i(j))$   $j = 0, 1, \dots, m$ . Then  $[\bar{F}_i(x)]^{1/x}$  is nonincreasing in  $x \geq 0$ . By the IFRA closure theorem

$$\begin{aligned} & \{[h[\bar{F}_1(j), \bar{F}_2(j), \dots, \bar{F}_n(j)]]\}^{1/j} \\ &= [h[q_{1j}(t), q_{2j}(t), \dots, q_{nj}(t)]]^{1/j} \end{aligned}$$

is nonincreasing in  $j \geq 0$  . ||

### Measures of Component Importance

Various measures of component importance have been suggested for components in a binary coherent system [6], [7]. These have a natural generalization to multi-state systems. For example, we say that component  $i$  is "critical" to a coherent system at state  $j$ , if, with component  $i$  in state  $j$ , the system is in state  $j$  and with component  $i$  *not* in state  $j$ , the system is *not* in state  $j$ . Let  $I_{ij}(\underline{p})$  be the probability of this event. We call  $I_{ij}(\underline{p})$  the *probability importance* of component  $i$  with respect to system state  $j$ .

To compute  $I_{ij}(\underline{p})$ , let

$$(1_{ij}, \underline{y}_j) = (y_{i0}, y_{i1}, \dots, y_{i,j-1}, 1, y_{i,j+1}, \dots, y_{im}) .$$

Then

$$\phi(1_{ij}, \underline{y}_j) - \phi(0_{ij}, \underline{y}_{j+1}) = 1$$

if and only if component  $i$  is critical to system states  $\{j, j+1, \dots, m\}$  in the sense that if component  $i$  is in state  $j$ , then the system is in one of the states  $\{j, j+1, \dots, m\}$  and if component  $i$  is *not* in state  $j$ , the system is *not* in one of the states  $\{j, j+1, \dots, m\}$ . Hence component  $i$  is critical to system  $j$  if and only if

$$[\phi(1_{ij}, \underline{y}_j) - \phi(0_{ij}, \underline{y}_j)] - [\phi(1_{i,j+1}, \underline{y}_{j+1}) - \phi(0_{i,j+1}, \underline{y}_{j+1})] = 1$$

for  $j = 1, 2, \dots, m-1$ . Hence



$$\begin{aligned}
 I_{ij}(\underline{p}) &= E[\phi(1_{ij}, \underline{Y}_j) - \phi(0_{ij}, \underline{Y}_j)] - E[\phi(1_{i,j+1}, \underline{Y}_{j+1}) - \phi(0_{i,j+1}, \underline{Y}_{j+1})], \\
 (2.7) \quad &= [h(1_{ij}, \underline{q}_j) - h(0_{ij}, \underline{q}_j)] - [h(1_{i,j+1}, \underline{q}_{j+1}) - h(0_{i,j+1}, \underline{q}_{j+1})]
 \end{aligned}$$

for  $j = 1, 2, \dots, m-1$ . It is easy to verify that

$$I_{im}(\underline{p}) = E[\phi(1_{im}, \underline{Y}_m) - \phi(0_{im}, \underline{Y}_m)]$$

and

$$I_{io}(\underline{p}) = E[\phi(1_{i1}, \underline{Y}_1) - \phi(0_{i1}, \underline{Y}_1)].$$

Note that for the case  $m = 1$ ,  $I_{io}(\underline{p}) = I_{im}(\underline{p})$ .

Finally, the probability that component  $i$  is in state  $j$  and component  $i$  is critical for system state  $j$  is  $p_{ij} I_{ij}(\underline{p})$ .

#### First Passage Distribution to State $j$

We now assume that components can "degrade" through successively lower states until, finally, total failure coincides with passage to state 0; i.e.

$$m \rightarrow m-1 \rightarrow \dots \rightarrow 2 \rightarrow 1 \rightarrow 0.$$

If state transition times are independent and exponentially distributed, this is called a "pure death process". In this section we do not allow transitions to higher states. Let  $F_{ij}$  be the (continuous) distribution of time until component  $i$  first reaches state  $j$ , starting in state  $m$  and  $\bar{F}_{ij}(t) = 1 - F_{ij}(t)$ . Let  $\zeta(t)$  be the state of the system at time  $t$ , then

$$P\{\zeta(s) > j; 0 \leq s \leq t\} = h[\bar{F}_{1j}(t), \bar{F}_{2j}(t), \dots, \bar{F}_{nj}(t)]$$

by (2.4) of Theorem 2.1.

We can also compute the probability that component  $i$  causes the system



to enter state  $j$ , in the sense that the system state changes from state  $j+1$  to state  $j$  simultaneously with a similar state transition by component  $i$ . If component first passage distributions to state  $j$  are continuous, as we assume, then at most one component can be responsible for the system changing state.

### 2.5 Proposition:

If first passage distributions of time to go from state  $m$  to state  $j$  ( $0 \leq j \leq m-1$ ),  $F_{ij}(t)$ , are continuous, then the probability that component  $i$  causes the system to pass to state  $j$  by time  $t$  is

$$\int_0^t \{ [h(1_{ij}, \bar{F}_j(u)) - h(o_{ij}, \bar{F}_j(u))] - [h(1_{i,j+1}, \bar{F}_{j+1}(u)) - h(o_{i,j+1}, \bar{F}_{j+1}(u))] \} dF_{ij}(u) .$$

where  $\bar{F}_j(u) = (\bar{F}_{1j}(u), \bar{F}_{2j}(u), \dots, \bar{F}_{nj}(u))$ .

### Proof:

By (2.7)

$$[h(1_{ij}, \bar{F}_j(u)) - h(o_{ij}, \bar{F}_j(u))] - [h(1_{i,j+1}, \bar{F}_{j+1}(u)) - h(o_{i,j+1}, \bar{F}_{j+1}(u))]$$

is the probability that component  $i$  is critical to system state  $j+1$  at time  $u$  in the sense that if component  $i$  is in state  $j+1$  at time  $u$ , the system is in state  $j+1$  at time  $u$ . Otherwise the system is not in state  $j+1$  at time  $u$ . If we multiply this by the probability,  $dF_{ij}(u)$ , that component  $i$  enters state  $j$  in the interval  $(u, u+du)$ ,

and integrate we have the required probability. Note that if component  $i$  is in state  $j + 1$  and critical to system state  $j + 1$  at time  $u$ , then it is a "worst" component in a "best" min path. Hence if component  $i$  then leaves state  $j + 1$  for state  $j$ , so does the system since component  $i$  was critical. ||

## REFERENCES

- [1] Barlow, R. E. and F. Proschan, STATISTICAL THEORY OF RELIABILITY AND LIFE TESTING, Holt, Rinehart and Winston, (1975).
- [2] Hirsch, W. M., M. Meisner and C. H. Boll, Cannibalization in Multi-Component Systems and the Theory of Reliability, Naval Research Logistics Quarterly, Vol. 15, No. 3, pp. 331-360, (1968).
- [3] Postelnicu, V., "Nondichotomic Multi-Component Structures," Bull. Math. de la Soc. Sci. Math. de la R. S. de Roumanie, Vol. 14 (62), No. 2, pp. 209-217, (1970).
- [4] Postelnicu, V., "Random Voting Rules," Centre of Mathematical Statistics Tech. Report, 21 Grivitei St., Bucharest 12, ROMANIA.
- [5] Butterworth, R. W., "A Set Theoretic Treatment of Coherent Systems," SIAM J. Appl. Math., Vol. 22, No. 9, (1972).
- [6] Barlow, R. E. and F. Proschan, "Importance of System Components and Fault Tree Events," Stochastic Processes and Their Applications, Vol. 3, pp. 1-21, (1975).
- [7] Lambert, H. E., "Measures of Importance of Events and Cut Sets in Fault Trees," Reliability and Fault Tree Analysis, SIAM, Philadelphia, pp. 77-100, (1975).